Approximately Permanent Electronic Orbits and the Origin of Spectral Series.

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The electrodynamical theory which we owe to Lorentz and Larmor provides theoretically a logical and consistent scheme whereby the equations of motion of electronic systems may be formulated. But, unfortunately, even most simple cases lead to equations of such complexity that the attempt to deduce exact solutions must at present be abandoned since there is no mathematical machinery available for the purpose. We have accordingly to make some simplifying assumptions, not strictly true, in order to obtain an approximate solution. In many cases, results are thus obtained which give a very close agreement with observation, and this is so far gratifying. But modern experimental work in radiation makes it clear that the phenomena have not yet been co-ordinated with the electrodynamical theory of electrons.

It is reasonable to enquire if this is due to the failure of mathematicians to provide an explanation, whether because the approximations used are not accurate enough, or because the conception of the electronic systems considered is not sufficiently general ?

As indicating failure due to the use of approximate equations, it may be pointed out that the neglect of radiation terms, or even their treatment by an approximation due to Larmor and Lorentz (valid in certain cases) is attended with considerable danger, and in former papers to the Royal Society I have given cases in which such approximate treatment would give a quite inadequate notion of the effects to be expected.

As regards the specification of an electronic system, it is remarkable that so many calculations take account of electrostatic forces only. In any attempt to deal with the phenomena of spectra it seems obvious that only a theory which takes account of magnetic and electromagnetic forces is likely to meet with success.

Hicks has on several occasions insisted on this, and recently* he has given an illustration showing how large the effect of internal magnetic force may be in affecting the motion of an electron.

Conway† has outlined a theory of spectral series, which appears to me most important. My investigations show that some vital modifications have to be introduced, and that we then get results of a very promising character in the explanation of series in accordance with pure electrodynamics.

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* 'Proc. Roy. Soc.,' vol. 90, p. 356 (1914).

† 'Phil. Mag.,' vol. 26, p. 1010 (1913).

A single electron carries a definite electric charge (whether of positive or negative sign) as a primary part of its constitution. It is also supposed to have a definite radius. Such an electron is capable of internal rotation as well as of translation, but comparatively little study has been made as to the consequences of rotation. It implies, of course, a magnetic moment, and some would thereby hope to explain magnetisation. I confess that I have no logical ground for sharing that hope, rather the reverse; and at present it seems necessary to retain a true permanent magnetic moment, as part of the constitution of an electron. We need not on that account omit the possibility of magnetic moment arising from rotation of the electron. It must, in fact, exist, although subject to variation of amount.

The complete formulation of the equations of motion of two such nuclei might be made, but it is a formidable undertaking. I accordingly take a more limited problem. Although it is thus only an illustration, the results are worthy of attention; and it may, like the artificial problem of a particle moving round a fixed centre of force, serve as a guide to real cases.

Corresponding to the atom, which is supposed to be comparatively massive, we shall take a nucleus of radius a, with a positive charge E, and a permanent magnetic moment μ. It is constrained to be fixed both as regards translation and rotation. The motion we propose to examine is that of a particle corresponding to a corpuscle with a negative charge $-e$. We shall assume that it has no permanent internal magnetic moment and that internal rotation is neglected. The speed of the particle is to be small enough to admit the use of its invariable mass m.

It will conduce to clearness if we begin by finding the motion which is possible when the radiation is neglected. Take the origin of co-ordinates at the centre of the positive nucleus and the z axis along the axis of magnetic moment. Further, let r, θ, ϕ be the polar co-ordinates of the moving corpuscle, and C the velocity of light.

On account of the electric and magnetic forces the motion is characterised by a Lagrangian function L, where

$$\frac{\mathrm{L}}{m} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + \frac{e\mu}{mr}\sin^2\theta\dot{\phi} + \frac{e\mathrm{E}\mathrm{C}^2}{mr},$$

when the corpuscle is outside the nucleus, from which we deduce an energy integral,

$$\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - \frac{e\mathrm{E}\mathrm{C}^2}{mr} = \tfrac{1}{2}v^2 \text{ (a constant)},$$

and an integral of angular momentum,

$$r^2 \sin^2 \theta \dot{\phi} + \frac{e\mu}{mr} \sin^2 \theta = h \text{ (a constant).}$$

The equations of motion, which we require to ascertain if orbits are stable are

$$\ddot{r} - r \sin^2 \theta \dot{\phi}^2 - r\dot{\theta}^2 + \frac{e\mu}{mr^2} \sin^2 \theta \dot{\phi} + \frac{e\mathrm{EC}^2}{mr^2} = 0.$$

$$\frac{d}{dt} r^2 \dot{\theta} - r^2 \sin \theta \cos \theta \dot{\phi}^2 - 2 \frac{e\mu}{mr} \sin \theta \cos \theta \dot{\phi} = 0,$$

$$\frac{d}{dt} \left\{ r^2 \sin^2 \theta \dot{\phi} + \frac{e\mu}{mr} \sin^2 \theta \right\} = 0.$$

The equations show that there may be steady motions in circular orbits. If, therefore, we assume

$$r = r_0, \qquad \theta = \alpha, \qquad \dot{\phi} = \omega,$$

for the constant values, we get, provided $\alpha \neq \frac{1}{2}\pi$,

$$\omega + \frac{2e\mu}{mr_0^3} = 0, \qquad \omega^2 - \frac{e\mu\omega}{mr_0^3} = \frac{e\mathrm{EC}^2}{mr_0^3 \sin^2 \alpha}.$$

Hence $$r_0^3 = -2\frac{e\mu}{m\omega}, \qquad \sin^2 \alpha = -\frac{1}{3}\frac{\mathrm{EC}^2}{\mu\omega}.$$

Thus such orbits exist provided ω is negative and numerically greater than $\frac{1}{3}\mathrm{EC}^2/\mu$, and if, further, r_0 is greater than a, the radius of the nucleus. This requires that $\frac{1}{6} m\mathrm{EC}^2 a^3/e\mu^2$ should be less than 1.

A disturbance from a steady state may be examined in the usual way by assuming that r, θ, $\dot{\phi}$, differ from their steady values by small quantities proportional to $e^{i\lambda t}$.

We find that

$$\begin{aligned} \lambda^2 &= \tfrac{1}{2}\omega^2 [(1 + \tfrac{9}{4} \sin^2 \alpha) \pm \{(1 + \tfrac{9}{4} \sin^2 \alpha)^2 - 3 \sin^2 2\alpha\}^{\frac{1}{2}}] \\ &\equiv \tfrac{1}{2}\omega^2 [(1 + \tfrac{9}{4} \sin^2 \alpha) \pm \{(1 - \tfrac{15}{4} \sin^2 \alpha)^2 + 3 \sin^4 \alpha\}^{\frac{1}{2}}]. \end{aligned}$$

Both values of λ^2 are positive, so that if the orbits exist they are stable.

When $\alpha = \frac{1}{2}\pi$ the equation is

$$\omega^2 - \frac{e\mu\omega}{mr_0^3} = \frac{e\mathrm{EC}^2}{mr_0^3}, \qquad \text{or} \qquad r_0^3 = \frac{e}{m} \cdot \frac{(\mathrm{EC}^2 + \mu\omega)}{\omega^2}.$$

For displacements in the plane of the orbit varying as $e^{i\lambda t}$ we get

$$\lambda^2 = \frac{\omega^2 \{\mathrm{EC}^2 + (\sqrt{3} - 1)\mu\omega\} \{\mathrm{EC}^2 - (\sqrt{3} + 1)\mu\omega\}}{(\mathrm{EC}^2 + \mu\omega)^2},$$

so that for stability this must be positive.

For displacements perpendicular to the plane of the orbit varying as $e^{i\lambda t}$ we get

$$\lambda^2 = \omega^2 \frac{(\mathrm{EC}^2+\mu\omega)(\mathrm{EC}^2+3\mu\omega)}{(\mathrm{EC}^2+\mu\omega)^2},$$

and for stability this must be positive.

Hence with ω positive the orbits are entirely stable, with r_0 ranging from $a\left\{(3\sqrt{3}+5)\frac{e\mu^2}{m\mathrm{EC}^2a^3}\right\}^{\frac{1}{3}}$ to ∞, provided $\frac{m\mathrm{EC}^2a^3}{e\mu^2} < (3\sqrt{3}+5)$. If this condition is not satisfied, r_0 ranges from a to ∞.

With ω negative, the orbits are entirely stable with r_0 ranging from $a\left(\frac{6e\mu^2}{m\mathrm{EC}^2a^3}\right)^{\frac{1}{3}}$ to ∞, provided $\frac{m\mathrm{EC}^2a^3}{e\mu^2} < 6$. If this condition is not satisfied, r_0 ranges from a to ∞.

So far we have assumed that the corpuscle does not penetrate within the nucleus of radius a. If it does, the equations change form, and we must make a further specification as to the interior. A usual hypothesis is that the charge is uniformly distributed throughout the sphere. I shall follow Conway in this supposition and further assume that the magnetic moment is uniformly distributed; but in anticipation of an important point which will occur later, I shall suppose that the dielectric ratio of the matter inside is given by K, where $\mathrm{K}^{\frac{1}{2}} = \mathrm{C}/\mathrm{C}'$.

In order to secure proper continuity with the equations outside the nucleus we find that the equations inside are:—

$$\ddot{r} - r\sin^2\theta\dot{\phi}^2 - r\dot{\theta}^2 - \frac{e\mu}{ma^5}r(5a^2-6r^2)\sin^2\theta\dot{\phi} + \frac{e\mathrm{EC}'^2}{ma^3}r = 0.$$

$$\frac{d}{dt}r^2\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 - \frac{e\mu}{ma^5}r^2(5a^2-3r^2)\sin\theta\cos\theta\dot{\phi} = 0.$$

$$\frac{d}{dt}\left\{r^2\sin^2\theta\dot{\phi} + \tfrac{1}{2}\frac{e\mu}{ma^5}r^2(5a^2-3r^2)\sin^2\theta\right\} = 0.$$

There is a Lagrangian function,

$$\mathrm{L} = \tfrac{1}{2}\{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\} + \tfrac{1}{2}\frac{e\mu}{ma^5}r^2(5a^2-3r^2)\sin^2\theta\dot{\phi} - \tfrac{1}{2}\frac{e\mathrm{EC}'^2}{ma^3}r^2,$$

from which we can deduce an energy integral and an integral of angular momentum.

There are steady circular orbits given by

$$r = r_0, \qquad \theta = \alpha, \qquad \dot{\phi} = \omega;$$

and if $\alpha \neq \frac{1}{2}\pi$ we get

$$\omega^2 + \frac{e\mu}{ma^5}\omega(5a^2-6r_0^2) = \frac{e\mathrm{EC}'^2}{ma^3\sin^2\alpha},$$

$$\omega + \frac{e\mu}{ma^5}(5a^2-3r_0^2) = 0.$$

Thence
$$\omega r_0^2 \sin^2 \alpha = -\tfrac{1}{3}\frac{\mathrm{EC}'^2 a^2}{\mu}.$$

Thus, as Conway points out, all such orbits if they exist have the same angular momentum.

For disturbance varying as $e^{i\lambda t}$ we get

$$\lambda^2 = \omega^2 \cos^2 \alpha + \left(\frac{e\mu}{ma^5}\right)^2 \sin^2 \alpha \, (6r_0^2 - 5a^2)^2 \quad \text{and} \quad \lambda^2 = 0,$$

so that the orbits are stable as regards one type of disturbance and neutral as regards the other.

The relation between r_0 and α is

$$\sin^2 \alpha = \frac{1}{9} \cdot \frac{m\mathrm{EC}'^2 a^3}{e\mu^2} \cdot \frac{1}{\{25/36 - (5/6 - r_0^2/a^2)^2\}}.$$

As r_0 ranges from a to $\left(\frac{5}{6}\right)^{\frac{1}{2}} a$, $\sin^2 \alpha$ ranges from $\frac{4}{24} \cdot \frac{m\mathrm{EC}'^2 a^3}{e\mu^2}$ to $\frac{4}{25} \cdot \frac{m\mathrm{EC}'^2 a^3}{e\mu^2}$.

$\mathrm{Sin}^2 \alpha$ then increases to 1,

while r_0 decreases from $\left(\frac{5}{6}\right)^{\frac{1}{2}} a$ to $\left[\frac{1}{6}\left\{5 - \left(25 - 4\frac{m\mathrm{EC}'^2 a^3}{e\mu^2}\right)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}}$.

Clearly the orbits are possible only if $\frac{4}{25} \cdot \frac{m\mathrm{EC}'^2 a^3}{e\mu^2} < 1$,

When $\alpha = \frac{1}{2}\pi$ we get the condition

$$\omega^2 + \frac{e\mu}{ma^5}\, \omega\, (5a^2 - 6r_0^2) = \frac{e\mathrm{EC}'^2}{ma^3},$$

or

$$\frac{r_0^2}{a^2} = \frac{5}{6} + \frac{1}{6} \cdot \frac{\omega^2 - e\mathrm{EC}'^2/ma^3}{\omega\, e\mu/ma^3}.$$

For disturbance in the plane varying as $e^{i\lambda t}$ we get

$$\lambda^2 = 4\omega\left\{\omega + \frac{e\mu}{ma^3}\left(5 - 3\frac{r_0^2}{a^2}\right)\right\} + \left(\frac{e\mu}{ma^3}\right)^2 \left(5 - 6\frac{r^2}{a^2}\right)^2.$$

For disturbance perpendicular to the plane

$$\lambda^2 = \omega\left\{\omega + \frac{e\mu}{ma^3}\left(5 - 3\frac{r_0^2}{a^2}\right)\right\}.$$

Thus stability is secured if the latter quantity is positive. The possible range is as follows:—

$r_0 = a$......... ω is positive and $= \frac{1}{2} \cdot \frac{e\mu}{ma^3} + \left\{\left(\frac{1}{2} \cdot \frac{e\mu}{ma^3}\right)^2 + \frac{e\mathrm{EC}'^2}{ma^3}\right\}^{\frac{1}{2}}$,

$r_0 = \left(\frac{5}{6}\right)^{\frac{1}{2}} a$ ... ω " " $= \left(\frac{e\mathrm{EC}'^2}{ma^3}\right)^{\frac{1}{2}}$,

$r_0 = 0$......... ω " " $= -\frac{5}{2} \cdot \frac{e\mu}{ma^3} + \left\{\left(\frac{5}{2} \cdot \frac{e\mu}{ma^3}\right)^2 + \frac{e\mathrm{EC}'^2}{ma^3}\right\}^{\frac{1}{2}}$,

$$r_0 = 0 \ldots\ldots\ldots \quad \omega \text{ is negative and } = -\frac{5}{2} \cdot \frac{e\mu}{ma^3} - \left\{\left(\frac{5}{2} \cdot \frac{e\mu}{ma^3}\right)^2 + \frac{e\mathrm{EC}'^2}{ma^3}\right\}^{\frac{1}{2}},$$

$$r_0 = \left(\frac{5}{6}\right)^{\frac{1}{2}} a \ldots \quad \omega \quad ,, \quad ,, \quad = -\left(\frac{e\mathrm{EC}'^2}{ma^3}\right)^{\frac{1}{2}},$$

$$r_0 = a \ldots\ldots\ldots \quad \omega \quad ,, \quad ,, \quad = \frac{1}{2} \cdot \frac{e\mu}{ma^3} - \left\{\left(\frac{1}{2} \cdot \frac{e\mu}{ma^3}\right)^2 + \frac{e\mathrm{EC}'^2}{ma^3}\right\}^{\frac{1}{2}}.$$

For positive values of ω the orbits are stable since $r_0 < a$, while for negative values of ω we must exclude the range for which ω lies between :—

$$-\frac{5}{2} \cdot \frac{e\mu}{ma^3} + \left\{\left(\frac{5}{2} \cdot \frac{e\mu}{ma^3}\right)^2 - \frac{e\mathrm{EC}'^2}{ma^3}\right\}^{\frac{1}{2}}$$

and

$$-\frac{5}{2} \cdot \frac{e\mu}{ma^3} - \left\{\left(\frac{5}{2} \cdot \frac{e\mu}{ma^3}\right)^2 - \frac{e\mathrm{EC}'^2}{ma^3}\right\}^{\frac{1}{2}}.$$

These results may usefully be made more general by assuming that the total magnetic moment μ consists of two parts, μ_1 and μ_2, where $\mu_1 + \mu_2 = \mu$. If the part μ_1 is uniformly disturbed throughout as before, while μ_2 is confined to the surface, the effect of the latter is to superpose a uniform field within the nucleus. The equations for the circular orbits of equal angular momentum are now

$$-\omega = \frac{5\,e\mu_1}{ma^3} - \frac{e\mu_2}{ma^3} - \frac{3\,e\mu_1}{ma^3} \cdot \frac{r_0^2}{a^2},$$

$$\omega r_0^2 \sin^2 \alpha = -\frac{1}{3} \cdot \frac{\mathrm{EC}'^2 a^2}{\mu_1}.$$

The general character of the orbits is not altered, but the necessity for this subdivision will become apparent when we attempt to fit our final results with observation.

It will now be convenient to consider the magnitude of the quantities involved, and for the moment it will suffice to take the order of magnitude only without entering on precise numerical estimates.

Thus let us take $\mathrm{E} = e = 10^{-20}$, $e/m = 10^7$, $\mu = 10^{-21}$, $a = 10^{-8}$, $\mathrm{C}^2 = 10^{21}$.

The quantity $\dfrac{m\mathrm{EC}^2 a^3}{e\mu^2}$ is thus of order 10^{12},

$\left(\dfrac{e\mathrm{EC}^2}{ma^3}\right)^{\frac{1}{2}}$,, 10^{16},

$\dfrac{e\mu}{ma^3}$,, 10^{10}.

Thus, if K is unity, it is clear that the stationary orbits with constant angular momentum discovered by Conway cannot exist for the given system, and although the equatorial orbits do exist, it will be noted that the frequency varies so little within the nucleus that the explanation of series on these lines would fail.

It would hardly be considered expedient to suppose that μ is any large multiple of the above value, and the alternative course of reducing a to a small enough value (of order 10^{-12}) to make the orbits possible, carries with it, as a consequence, that the angular momentum in the orbits is but a small fraction of the value known as Planck's unit. Nevertheless, the theory outlined by Conway gives such good promise that a method of escape from the difficulty just raised is worthy of consideration.

We may suppose that the charge of the nucleus is not uniformly distributed, but is chiefly concentrated near the surface. This hypothesis, or, as an alternative, the hypothesis that the material has a large value of K, would enormously reduce the electrostatic forces inside the nucleus and so make the orbits possible. We can still obtain frequencies of the proper order, and, moreover, now with a suitable range by taking a of order 10^{-10}. But it must be carefully noted that the angular momentum now has a value very small compared with Planck's value. As already stated, I think we are required to retain the view that the magnetic moment of the nucleus is permanent and does not arise from rotation. It is also desirable and permissible to retain 10^{-8} as the effective radius for external purposes, and to regard 10^{-10} as the radius of the region within which the orbits are possible, whether as a consequence of small electrical density or of a large value of K.

The form of the meridian section of the surface on which the particles with the same angular momentum lie is of interest. Taking the simpler case in which μ is undivided, the curve consists of two branches with a cusp at $r_0 = (\frac{5}{6})^{\frac{1}{2}} a$. If $\frac{4}{25}(m\text{EC}'^2 a^3/e\mu^2)$ is just a little less than unity, the orbits are rather close to the equatorial plane and within a narrow range $r_0 = 0{\cdot}9a$ to a. The values of ω, which are negative, range from two to two and a half times $e\mu/ma^3$, which is hardly enough. On the other hand, if $\frac{4}{25}(m\text{EC}'^2 a^3/e\mu^2)$ is small the orbits lie rather close to the magnetic axis of the nucleus and range from $r_0 = 0$ to a while ω ranges from two to five times $e\mu/ma^3$, which is more suitable. The frequency of a disturbance is not in general $\omega/2\pi$, but on the latter assumption the difference is small. Anticipating a result which will appear, when we take account of the radiation, that only for certain series of values of r_0/a are the orbits approximately permanent, we get a promising explanation of certain spectral series. But there is one weakness. For a given value of r_0 there is but one frequency of disturbance. Thus the series of lines can have no structure, although they probably have general diffuseness.

The equatorial orbits, although they have not all the same angular momentum, may supply the deficiency. Corresponding to a given value of r_0 there are two values of ω, and, within certain ranges, both are stable. Further, for each value of ω there are two frequencies of disturbance. Thus the series may be expected to give fourfold structure. The formulæ in this case cannot, in general, be put in the simple form which obtains for the orbits of same angular momentum, and which agrees so closely with the empirical formulæ. Within certain limits, however, approximate formulæ of the empirical type would be valid.

We now proceed to consider the effects to be expected on account of radiation from the particle. Since it moves in a region of electric and magnetic force in phase with its own, the usual assumption that it radiates proportionally to the square of its resultant acceleration is not applicable. The forces arising from the radiation going on both inside and outside the nucleus entirely modify the problem. It appears that certain of the stationary orbits are picked out in series as having definite frequencies, and for these the radiation is comparatively small. These orbits may then continue to exist for a long time (optically), and so manifest their existence in the spectrum. The intermediate orbits are not permanent and the radiation is considerable. They must, however, take part in the process by which the radiation from the more permanent orbits is maintained, and some portion of their radiation will be exhibited outside the nucleus as Röntgen radiation of short wave-length (determined by the outer radius of the nucleus).

These results may be deduced from the fundamental equations.

It may be recalled that Lamb* investigated the behaviour of a fixed sphere of atomic dimensions $a = 10^{-8}$ with a large dielectric ratio. He shows that series of vibrations of optical frequencies may arise and that the damping is very slight. Although Lamb's series are not in agreement with the empirical formulæ for observed series, I have for some time entertained the view that some simple modification of the vibrating system would give the empirical forms. In 'Phil. Trans.,' 1910, I obtained extensions of Lamb's equations and showed that, in addition to the optical vibrations, there was one rapidly damped vibration of wave-length comparable with the circumference of the sphere $2\pi 10^{-8}$. When the paper was written we had no knowledge of the wave-length of Röntgen radiation, but it is known now that it is of this order.

Returning, now, to the problem before us, the motion of the particle will set up radiation both inside and outside the nucleus. That radiation, as well as the exciting radiation, can be represented in terms of disturbances of electric and magnetic type, arising from harmonics of different orders. These produce reactions on the moving particle. Complete formulation might be made but is hardly yet justified. It will be simpler and clearer to confine attention to the first-order terms.

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* 'Camb. Phil. Trans.,' Stokes Commemoration Volume, 1899.

Take first the first-order disturbance of electric type with axis along the magnetic axis of the nucleus. The expressions for the electric and magnetic force inside are (*loc. cit. ante*, p. 174)—

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \frac{\mathrm{C}'}{r^3}(0, 0, -1)\,\{r^2(\psi_1''+\psi_2'')+r(\psi_1'-\psi_2')+\psi_1+\psi_2\}$$

$$+\frac{\mathrm{C}'z}{r^5}(x, y, z)\,\{r^2(\psi_1''+\psi_2'')+3r(\psi_1'-\psi_2')+3(\psi_1+\psi_2)\}.$$

$$\mathrm{K}^{-\frac{1}{2}}(\alpha, \beta, \gamma) = \frac{\mathrm{C}'}{r^3}(-y, x, 0)\,\{r(\psi_1''-\psi_2'')+\psi_1'+\psi_2'\},$$

where $\psi_1 = \psi_1(\mathrm{C}'t-r)$, $\psi_2 = \psi_2(\mathrm{C}'t+r)$, and $\psi_1(\mathrm{C}'t)+\psi_2(\mathrm{C}'t) = 0$.

Since K is to be taken as very large, the reaction on the particle depends almost entirely on the dynamic force arising from (α, β, γ) on the velocity $(\dot{x}, \dot{y}, \dot{z})$, since the electric force is small in comparison.

Thus the additional terms to be introduced in the equations of motion of the particle vary as $\{r(\psi_1''-\psi_2'')+\psi_1'+\psi_2'\}$. These will vanish and the stationary orbit will be permanent for that type of disturbance if r is selected so that

$$r(\psi_1''-\psi_2'')+\psi_1+\psi = 0.$$

Hence, taking $\psi_1 = \mathrm{A}e^{i\mathrm{K}^{\frac{1}{2}}\lambda(\mathrm{C}'t-r)/a}$, $\psi_2 = -\mathrm{A}e^{i\mathrm{K}^{\frac{1}{2}}\lambda(\mathrm{C}'t+r)/a}$, we get the condition

$$\tan \mathrm{K}^{\frac{1}{2}}\lambda r/a = \mathrm{K}^{\frac{1}{2}}\lambda r/a.$$

The roots are

$$\frac{1}{\pi}\mathrm{K}^{\frac{1}{2}}\lambda r/a = 1{\cdot}4303,\ 2{\cdot}4590,\ 3{\cdot}4709,\ 4{\cdot}4774,\ 5{\cdot}4818,\ 6{\cdot}4844, \text{ etc.},$$

and the higher roots are nearly

$$(n+\tfrac{1}{2})-\frac{1}{\pi^2}(n+\tfrac{1}{2})^{-1}.$$

In order to satisfy the boundary conditions at the surface of the nucleus, $\mathrm{K}^{\frac{1}{2}}\lambda$ is determined by

$$\tan \mathrm{K}^{\frac{1}{2}}\lambda = \mathrm{K}^{\frac{1}{2}}\lambda\left\{1-\frac{\mathrm{K}\lambda^2(1-i\lambda)}{(\mathrm{K}-1)(1-i\lambda)-i\mathrm{K}\lambda^3}\right\},$$

Since K is large, the real parts of the roots are very nearly those of $\tan \mathrm{K}^{\frac{1}{2}}\lambda = \mathrm{K}^{\frac{1}{2}}\lambda$, while the imaginary parts are very small. There is also the special root $\lambda = \frac{i}{2}\pm\frac{1}{2}\sqrt{3}$.

Thus $\frac{1}{\pi}\mathrm{K}^{\frac{1}{2}}\lambda$ has the values already given for $\frac{1}{\pi}\mathrm{K}^{\frac{1}{2}}\lambda\frac{r}{a}$. Hence r/a may be given series of values, viz.,

$$r/a = 1, \quad \frac{1{\cdot}4303}{2{\cdot}4590}, \quad \text{etc.,} \quad \frac{1{\cdot}4303}{n+\frac{1}{2}}, \ \ldots\ldots \ 0;$$

$$r/a = 1, \quad \frac{2{\cdot}4590}{3{\cdot}4709}, \quad \text{etc.,} \quad \frac{2{\cdot}4590}{n+\frac{1}{2}}, \ \ldots\ldots \ 0;$$

and so on.

Our argument, then, is that at these distances the corresponding orbits are for optical purposes of considerable permanence for disturbance of this type, and so give rise to series, while intermediate orbits, being soon deranged by this type of disturbance, give no optical effect.

The orbits of equal angular momentum are given by

$$-\omega = -\frac{e\mu_2}{ma^3}+\frac{5e\mu_1}{ma^3}\left(1-\frac{3r_0^2}{a^2}\right),$$

where $\mu_2+\mu_1 = \mu$ and r/a may be given the above series of values. Thus, if the conditions are such that the orbits lie close to the axis, the frequencies of the lines in the series are

$$\nu = -\frac{1}{2\pi}\omega = -\frac{1}{2\pi}\cdot\frac{e\mu_2}{ma^3}+\frac{5}{2\pi}\cdot\frac{e\mu_1}{ma_3}\left(1-\frac{3}{5}\cdot\frac{r_0^2}{a^2}\right).$$

The head is $\nu = \frac{1}{\pi}\cdot\frac{e\mu_1}{ma_3}-\frac{1}{2\pi}\cdot\frac{e\mu_2}{ma_3}$ and the tail is $\nu = \frac{5}{2\pi}\cdot\frac{e\mu_1}{ma^3}-\frac{1}{2\pi}\cdot\frac{e\mu_2}{ma^3}$.

Towards the tail the values will be approximately

$$\nu = -\frac{1}{2\pi}\cdot\frac{e\mu_2}{ma^3}+\frac{5}{2\pi}\cdot\frac{e\mu_1}{ma^3}\left\{1-\frac{\frac{3}{5}\mathrm{S}}{(n+\frac{1}{2})^2}\right\},$$

where S for the different series has the values

$$(1{\cdot}4303)^2, \ (2{\cdot}4590)^2, \ \text{etc.}$$

Near the head the more exact values must be introduced in place of $(n+\frac{1}{2})^2$.

If, however, the orbits approach the equatorial plane, and we consider also equatorial orbits which are not all of the same angular momentum, the formula, although having the same general features, cannot be accurately expressed in so simple a form, and, moreover, limits are set to the possible number of lines.

We next consider the first-order disturbance of magnetic type. The field is given (as at *loc. cit. ante*, p. 187) by

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \frac{\mathrm{C}'}{r^3}(y, -x, 0)\,\{r(\psi_1''-\psi_2'')+\psi_1'+\psi_2'\},$$

$$\mathrm{K}^{-\frac{1}{2}}(\alpha, \beta, \gamma) = \frac{\mathrm{C}'}{r^3}(0, 0, -1)\,\{r^2(\psi_1''+\psi_2'')+r(\psi_1'-\psi_2')+\psi_1+\psi_2\}$$
$$+\frac{\mathrm{C}'z}{r^5}(x, y, z)\,\{r^2(\psi_1''+\psi_2'')+3r(\psi_1'-\psi_2')+3(\psi_1+\psi_2)\}.$$

As before, the reactions on the particle arise chiefly from α, β, γ, and are found to consist of terms proportional to

$$r(\psi_1' - \psi_2') + \psi_1 + \psi_2$$

and to

$$r^2(\psi_1'' + \psi_2'') + r(\psi_1' - \psi_2') + \psi_1 + \psi_2.$$

They will vanish only if

$$\tan \mathrm{K}^{\frac{1}{2}}\lambda r/a = \mathrm{K}^{\frac{1}{2}}\lambda r/a \quad \text{and} \quad \tan \mathrm{K}^{\frac{1}{2}}\lambda r/a = \frac{\mathrm{K}^{\frac{1}{2}}\lambda r/a}{1 - \mathrm{K}\lambda^2 r^2/a^2}.$$

We cannot in general satisfy both with the same value of r, but it may be noted that for orbits near the axis it is more important to satisfy the first condition, while for orbits near the equatorial plane the more important condition is the second. We may therefore expect approximately permanent orbits. If they are near the axis we have the series

$$\frac{1}{\pi}\mathrm{K}^{\frac{1}{2}}\lambda r/a = 1{\cdot}4303,\ 2{\cdot}4590,\ \ldots,\ n + \tfrac{1}{2} - \frac{1}{\pi^2}(n + \tfrac{1}{2})^{-1},$$

while, if they are near the equatorial plane, we have

$$\frac{1}{\pi}\mathrm{K}^{\frac{1}{2}}\lambda r/a = 0{\cdot}87334,\ 1{\cdot}9470,\ \ldots,\ n - \frac{1}{\pi^2}n^{-1}.$$

In either case the values of $\mathrm{K}^{\frac{1}{2}}\lambda$ are determined by the equation

$$\tan \mathrm{K}^{\frac{1}{2}}\lambda = \frac{\mathrm{K}^{\frac{1}{2}}\lambda}{1 - \mathrm{K}(1 - i\lambda)},$$

and, since K is large, the roots are practically

$$\frac{1}{\pi}\mathrm{K}^{\frac{1}{2}}\lambda = 1, 2, 3, \ldots, n,$$

while the imaginary part is small.

There is the special root $\lambda = i$ which corresponds to a rapidly damped disturbance of exponential type.

We thus expect approximately permanent orbits, those near the axis being given by

$$\frac{r}{a} = \frac{1{\cdot}4303}{n}, \text{ where } n \text{ is an integer not less than } 2,$$

$$\frac{r}{a} = \frac{2{\cdot}4590}{n}, \quad \text{,,} \qquad \text{,,} \qquad \text{,,} \qquad 3,$$

and so on.

On the other hand, if they are near the equatorial plane, r/a takes the values

$$\frac{r}{a} = \frac{0{\cdot}87334}{n}, \text{ where } n \text{ may now begin at } 1,$$

$$\frac{r}{a} = \frac{1{\cdot}97470}{n}, \quad \text{,,} \qquad \text{,,} \qquad 2,$$

and so on.

Corresponding to these we have, if the orbits are close to the axis, series of the usual Balmer type. The first series is given by frequency,

$$\nu = -\frac{1}{2\pi}\omega = -\frac{1}{2\pi}\cdot\frac{e\mu_2}{ma^3}+\frac{5}{2\pi}\cdot\frac{e\mu_1}{ma^3}\left\{1-\frac{3}{5}\cdot\frac{(1{\cdot}4303)^2}{n^2}\right\}.$$

For those near the equatorial plane we should have to use the expressions involving square roots, and limits are set to the number of lines.

We may expect these series of magnetic type to be more diffuse than those of electric type. It is, of course, obvious that disturbances of higher order may arise, and that the series only have approximate permanence, since those selected as permanent for disturbance of electric type are not permanent for disturbance of magnetic type and *vice versâ*. This, however, suggests a basis of explanation for the presence or absence of certain series according to the source of excitation. And it may be noticed that while the two types of series cannot exist together, they might both be exhibited photographically if the exciting disturbance changes at optically long intervals from one type to the other.

We may now compare with observed results the simpler formulæ which hold when the orbits lie near the axis. They appear to have a direct application to certain cases, although we ought not to lose sight of the more complex expressions arising from equatorial orbits, which probably have a wider application.

We have, then, the various series of Balmer type, of which the first is given by

$$\nu = -\frac{1}{2\pi}\cdot\frac{e\mu_2}{ma^3}+\frac{5}{2\pi}\cdot\frac{e\mu_1}{ma^3}\left\{1-\frac{3}{5}\cdot\frac{(1{\cdot}4303)^2}{n^2}\right\} \quad (n = 2, 3, 4, \text{etc.}),$$

and the series of more general Rydberg type,

$$\nu = -\frac{1}{2\pi}\cdot\frac{e\mu_1}{ma^3}+\frac{5}{2\pi}\cdot\frac{e\mu_2}{ma^3}\left\{1-\frac{3}{5}\cdot\frac{(1{\cdot}4303)^2}{p^2}\right\},$$

in which p takes the values 1·4303, 2·4590, etc., while the higher values approximate to the well-known form $n+\frac{1}{2}$, where n is an integer. It may prove crucial that the lower values differ considerably from $n+\frac{1}{2}$.

For general exploratory purposes it may be useful to note that the numbers,

$$(1{\cdot}4303)^2,\ (2{\cdot}4590)^2,\ \text{etc.},$$

are very nearly the whole numbers

$$2,\quad 6,\quad 12,\quad 20,\quad 30,\quad 42,\quad \ldots,\quad n(n+1),$$

although for exact comparison these would not be sufficiently accurate.

If we write $\mu_1 = (1-k)\mu$, $\mu_2 = k\mu$, where $k < 1$, we find that the tail of both the Balmer and Rydberg series is the same, viz., $\nu_{\infty} = \frac{5}{2\pi} \cdot \frac{e\mu_1}{ma^3} \cdot \frac{(5-6k)}{5(1-k)}$. and since we cannot admit a negative value k must be $< \frac{5}{6}$. When the value of k is determined, we must take the heads of the series (in general different) as the last one which keeps ν positive. For a positive value of ω means that the corresponding orbit of the same angular momentum is impossible. Its place is taken by a corresponding extension of the region of possible equatorial orbits. It would seem fairest to suppose that our model corresponds to the elementary hydrogen atom.

In his recent Bakerian lecture, and in a private letter to me, Prof. Fowler now entertains the view that the hydrogen spectrum contains only series expressed by Balmer's formula, and that the extensive series of lines formerly* attributed to hydrogen are better represented by an extended Balmer formula, and do not represent hydrogen. I adopt his view, as I feel that this matter of arranging lines in series is one which can only be undertaken with advantage by those having intimate personal experimental knowledge of the spectra.

Rydberg's constant in wave numbers is 109675, which we may call N. The question now arises, which of the series of Balmer type in our formula is the one which gives N? That is to say, shall we take $\frac{5}{2\pi} \cdot \frac{e\mu_1}{ma^3}$ as $\frac{5}{3(1{\cdot}4303)^2} \mathrm{N} \,.\, 3 \times 10^{10}$, $\frac{5}{3(2{\cdot}4590)^2} \mathrm{N} \,.\, 3 \times 10^{10}$, or etc.? It makes no difference to the sequence of differences obtained or to the tail of the series, and the only effect on the principal series is to strike out in succession the lines at the head.

Thus, if we take $\frac{5}{2\pi} \cdot \frac{e\mu_1}{ma^3} = \frac{5\mathrm{N}}{3(1{\cdot}4303)^2} \times 3 \times 10^{10} = 89352 \times 3 \times 10^{10}$, and try to represent the experimental wave numbers 15237, 20570, 23038, ..., 27423, we get $\frac{5-6k}{5(1-k)} \times 89352 = 27423$, or $k = \frac{7}{9}$ nearly, and the principal Balmer series is

$n = 2.$	$n = 3.$	$n = 4.$	$n = 5.$		$n = \infty.$
4,	15239,	20568,	23036,	...,	27423.

If we take

$$\frac{5}{2\pi} \cdot \frac{e\mu_1}{ma^3} = \frac{5\mathrm{N}}{3(2{\cdot}4590)^2} \times 3 \times 10^{10} = 30230 \times 3 \times 10^{10},$$

we have $$\frac{5-6k}{5(1-k)} \times 30230 = 27423, \quad \text{or} \quad k = \frac{1}{3}, \text{ nearly,}$$

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* 'Monthly Notices, R.A.S.,' vol. 73, pp. 67 and 68.

and the principal Balmer series is exactly as before with the first member 4 struck out. I cannot guarantee the accuracy of the fifth figure.

If our view is correct, we ought to expect that, under certain circumstances, there should be a series approximating to the Rydberg type. Whichever of the two cases above we select we get the following values :—

9285, 18319, 21952, 23773, 24815, ..., 27423.

Subsidiary series of both types are to be expected and may be computed. These differ according to the assumption as to the principal series, so that discrimination may be possible ; but it is clearly more important to settle the question of the first series of Rydberg type.

If now we take $e/m = 1{\cdot}7 \times 10^{7}$, $\mu = 1{\cdot}8 \times 10^{-21}$, the first supposition as to the Rydberg constant gives $a = 1{\cdot}2 \times 10^{-10}$, and the second $a = 2{\cdot}6 \times 10^{-10}$, the distribution of μ differing in the two cases. The angular momentum of the particle cannot exceed 4×10^{-5} of Planck's unit $6{\cdot}8 \times 10^{-27}$.

We may summarise the results by saying that within a nucleus possessing the usual positive electronic charge and possessing fixed magnetic moment of the value attributed to the magneton, we may have approximately permanent orbits of corpuscles with the usual value of e/m which give rise to series of spectral lines of recognised type. They are probably diffuse and have no structure. Equatorial orbits also give rise to series of the same general character. These are not expressible in so concise a form ; the number of lines is limited and may be expected to possess a fourfold structure. Here we may consider the question of the Zeeman effect. The form of equation for the equatorial orbits shows at once that the Zeeman effect of the usual type may be expected to arise. But the orbits of the same angular momentum, although affected by an external magnetic field, do not give structural Zeeman effect, but only an additional general diffuseness.

The radius of this nucleus required to give suitable optical values is of order 10^{-10}, which is considerably smaller than the usual estimate of the atomic radius.

The wave-length of the rapidly damped disturbance communicated by this nucleus to the exterior is $4\pi a/\sqrt{3}$, which is rather smaller than the value assigned to characteristic Röntgen radiation. It is, however, not a little curious that the wave-length of the radiation from a corpuscle just outside the nucleus comes out about 2×10^{-8}, and this is sufficiently near observed values.

The effective radius of an atom for external purposes is, as we have said, usually estimated at about 10^{-8}.

Our results would not be affected if there is an outer shell extending to this distance, provided the inner nucleus is admitted. But one would prefer to have only one radius in specifying the atom. The value 10^{-8} depends largely on the kinetic theory of gases for hard spheres, and is admittedly rather weak. It is conceivable that some explanation is possible depending on the range of possible external corpuscular orbits at a given temperature. Meanwhile it is a point requiring consideration.

It may be added that, when the nucleus is free to move, corrections will be introduced in the formulæ we have obtained which depend on the mass of the nucleus.

Sir Joseph Larmor has very kindly discussed these results with me, and has suggested several important improvements of statement that have been adopted.

Electromagnetic Waves in a Perfectly Conducting Tube.

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(Communicated by Prof. A. W. Porter, F.R.S. Received December 5, 1914.)

The problem of waves in conducting tubes has already been treated by various authors.* Nevertheless, the solutions here proposed, offering certain peculiarities with respect to the velocity of the corresponding waves, and partly also with respect to the shape and distribution of the lines of force, seemed worthy of notice.

Let z be measured along the axis of an infinite right cylindrical tube of circular section. The material of the tube is assumed to be a perfect conductor, its interior being empty or filled with air. The electromagnetic waves will be assumed throughout to be axially symmetrical and of permanent type, *i.e.* to conserve all their features while proceeding along the tube.

If r be the distance of a point from the z axis, further, R and Z the

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* J. J. Thomson, 'Recent Researches in Electricity and Magnetism,' 1893; Lord Rayleigh, 'Phil. Mag.,' vol. 43, p. 125 (1897); R. H. Weber, 'Ann. der Physik,' vol. 8, p. 721 (1902). An account of the experimental investigations by v. Lang, Drude and Becker will be found in a paper by A. Kalähne, 'Ann. der Physik,' vol. 18, p. 92 (1905). Kalähne's own theoretical investigations concern only a ring-shaped tube.